Equations for Planar Sundials

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1. Purpose

Equations necessary to make planar sundials are derived. You can choose any orientation for the plane of a sundial.

2. Reference

[1] Toshimi Taki, "Matrix Method for Coordinates Transformation," Rev. E, February 29, 2004. in Taki's Homepage(http://www.asahi-net.or.jp/~zs3t-tk/)

3. Symbols

- Obliquity of Ecliptic: ε
- Latitude: ϕ
- Hour Angle of the Sun: H
- Declination of the Sun: δ
- Azimuth: A
- Altitude: h
- Height of Nodus from Plane of Sundial: l
- Distance between Nodus G_0 and Nodus G_1: d
- Direction Cosine: (L, M, N)

4. Altitude and Azimuth of the Sun

4. Position of the Sun

Let the declination and hour angle of the Sun at any given time, δ and H, respectively. Declination should be between $-\varepsilon$ and $+\varepsilon$ (obliquity of the ecliptic). This coordinates can be expressed with the direction cosine in the coordinate system $O\text{-}x_e'\text{-}y_e'$ as follows.

$$\begin{pmatrix} L_{Se}' \\ M_{Se}' \\ N_{Se}' \end{pmatrix} = \begin{pmatrix} \cos\delta\cos(-H) \\ \cos\delta\sin(-H) \\ \sin\delta \end{pmatrix}$$

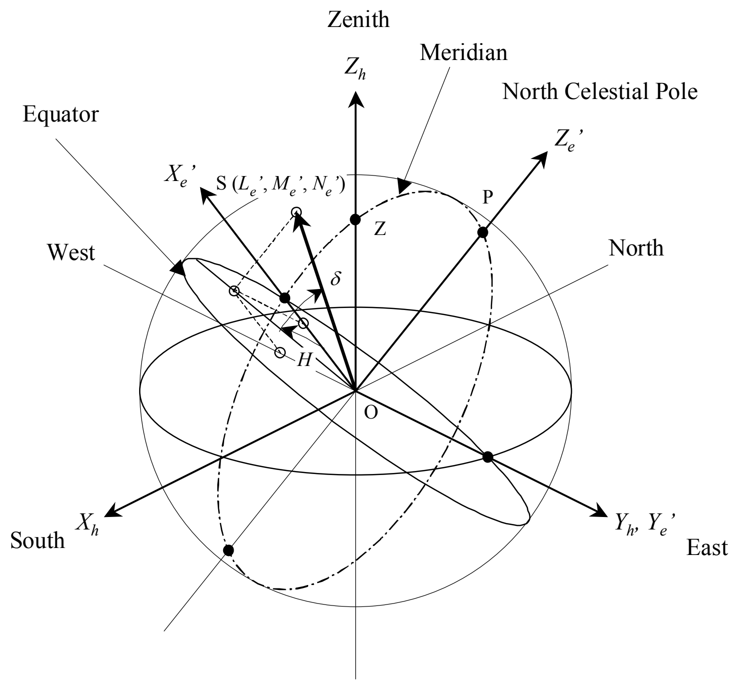

Figure 1. Equatorial Coordinate System

The position of the Sun is also expressed in azimuth A and altitude h. Azimuth is measured from the south to the west. The direction cosine of the Sun's position in horizontal coordinate system $O\text{-}x_h\text{-}y_h$ is expressed in the following equation.

$$\begin{pmatrix} L_{Sh} \\ M_{Sh} \\ N_{Sh} \end{pmatrix} = \begin{pmatrix} \cos h \cos(-A) \\ \cos h \sin(-A) \\ \sin h \end{pmatrix}$$

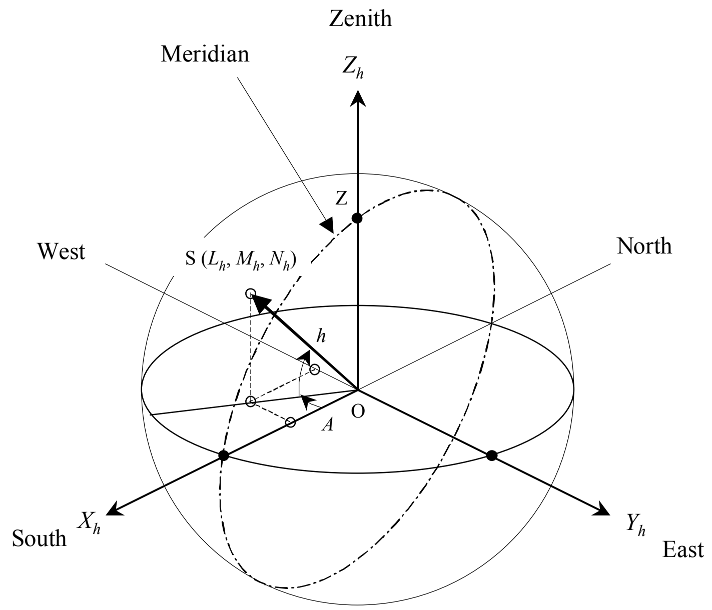

Figure 2. Horizontal Coordinate System

The equatorial coordinates of the Sun can be transformed to the horizontal coordinates using the following equation. The geographical latitude of the sundial is ϕ.

$$\begin{pmatrix} L_{Sh} \\ M_{Sh} \\ N_{Sh} \end{pmatrix} = \begin{bmatrix} \cos\left(\phi - \dfrac{\pi}{2}\right) & 0 & \sin\left(\phi - \dfrac{\pi}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(\phi - \dfrac{\pi}{2}\right) & 0 & \cos\left(\phi - \dfrac{\pi}{2}\right) \end{bmatrix} \begin{pmatrix} L_{Se}' \\ M_{Se}' \\ N_{Se}' \end{pmatrix}$$

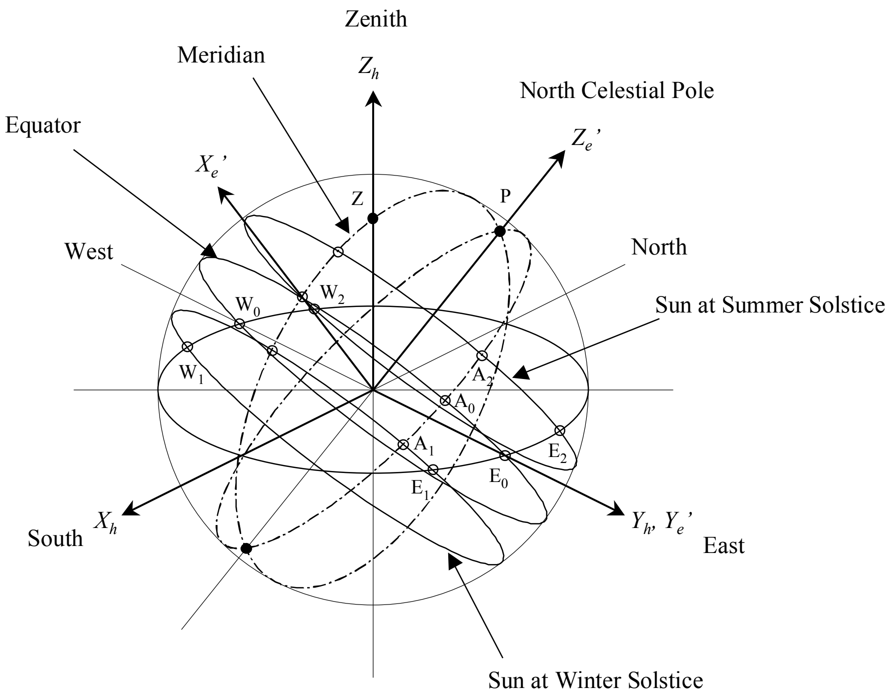

Figure 3. Position of the Sun

5. Position of Shadows

5.1 Plane of Sundial and Nodus

As shown in figure 3, the dial face of the sundial is in O-x''-y'' plane, where the coordinate system of the sundial is O-x''-y''. Main nodus G0 is set on z''-axis normal to the dial face with the distance l. Another nodus G1 is on the line directed to the north celestial pole from G0 and distance between G0 and G1 is d.

The relationship between the horizontal coordinate system and the sundial coordinate system is as follows. The horizontal coordinate system is rotated θ_x around Z_h-axis and becomes O-x'-y-z' coordinate system. Then, O-x'-y-z' coordinate system is rotated θ_y around y'-axis to the sundial coordinate system O-x''-y''-z''.

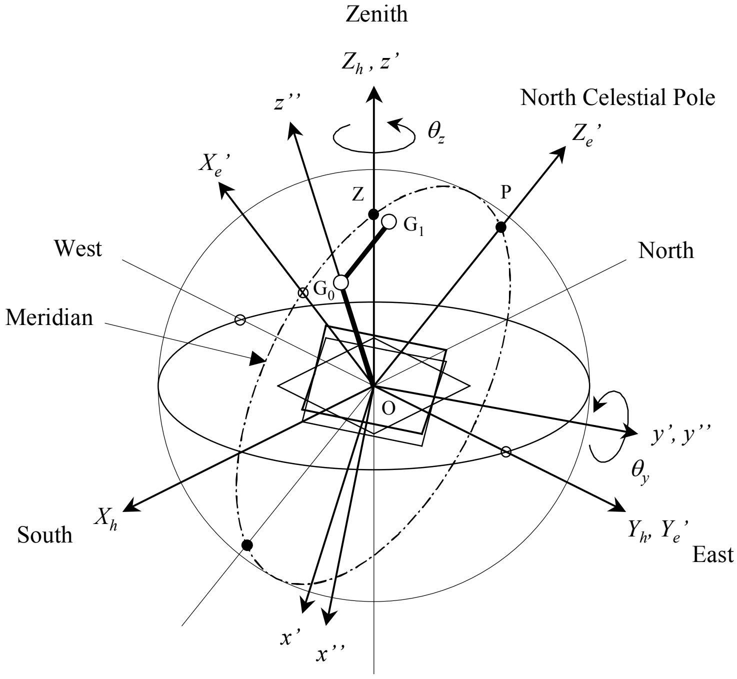

Figure 4. Sundial Coordinate System

The direction cosine in O-*x'*-*y'*-*z'* coordinate system is expressed as follows.

$$\begin{pmatrix} L' \\ M' \\ N' \end{pmatrix} = \begin{pmatrix} \cos\zeta'\cos\xi' \\ \cos\zeta'\sin\xi' \\ \sin\zeta' \end{pmatrix}$$

Then the direction cosine in the horizontal coordinates system is transformed to O-*x'*-*y'*-*z'* coordinate system.

$$\begin{pmatrix} L' \\ M' \\ N' \end{pmatrix} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} L_h \\ M_h \\ N_h \end{pmatrix}$$

Direction cosine in O-*x''*-*y''*-*z''* coordinate system is expressed as follows.

$$\begin{pmatrix} L'' \\ M'' \\ N'' \end{pmatrix} = \begin{pmatrix} \cos\zeta''\cos\xi'' \\ \cos\zeta''\sin\xi'' \\ \sin\zeta'' \end{pmatrix}$$

Finally, the direction cosine in O-*x'*-*y'*-*z'* coordinate system is transformed to the sundial coordinate system, O-*x''*-*y''*-*z''* coordinate system.

$$\begin{pmatrix} L'' \\ M'' \\ N'' \end{pmatrix} = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{pmatrix} L' \\ M' \\ N' \end{pmatrix}$$

Using the above equations, the direction cosine of the Sun in the horizontal coordinate system can be transformed to the direction cosine in the sundial coordinate system, (L_{sh}'', M_{sh}'', N_{sh}'').

5.2 Coordinates of Nodus G1 in Sundial Coordinate System

Direction cosine of Z_e'-axis in the equatorial coordinate system is (0, 0, 1). This vector is transformed to the sundial coordinate system as follows.

$$\begin{pmatrix} L_{Nh} \\ M_{Nh} \\ N_{Nh} \end{pmatrix} = \begin{bmatrix} \cos\left(\phi - \frac{\pi}{2}\right) & 0 & \sin\left(\phi - \frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(\phi - \frac{\pi}{2}\right) & 0 & \cos\left(\phi - \frac{\pi}{2}\right) \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} L_N' \\ M_N' \\ N_N' \end{pmatrix} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} L_{Nh} \\ M_{Nh} \\ N_{Nh} \end{pmatrix}$$

$$\begin{pmatrix} L_N'' \\ M_N'' \\ N_N'' \end{pmatrix} = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{pmatrix} L_N' \\ M_N' \\ N_N' \end{pmatrix}$$

Then, coordinates of nodus G1 in the sundial coordinate system is expressed as follows.

$$\begin{pmatrix} x_{G1}'' \\ y_{G1}'' \\ z_{G1}'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix} + d \begin{pmatrix} L_N'' \\ M_N'' \\ N_N'' \end{pmatrix}$$

5.3 Shadows of Nodus

Figure 5 shows shadows of the nodus.

The position of the shadow of nodus G0 in the sundial coordinate system is,

$$\begin{pmatrix} x_{GS0}'' \\ y_{GS0}'' \\ z_{GS0}'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix} - \frac{l}{N_S''} \begin{pmatrix} L_S'' \\ M_S'' \\ N_S'' \end{pmatrix}$$

The position of the shadow of nodus G1 in the sundial coordinate system is,

$$\begin{pmatrix} x_{GS1}'' \\ y_{GS1}'' \\ z_{GS1}'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ N_{G1}'' \end{pmatrix} - \frac{N_{G1}''}{N_S''} \begin{pmatrix} L_S'' \\ M_S'' \\ N_S'' \end{pmatrix}$$

It is obvious that z_{GS0}'' and z_{GS1}'' are zero.

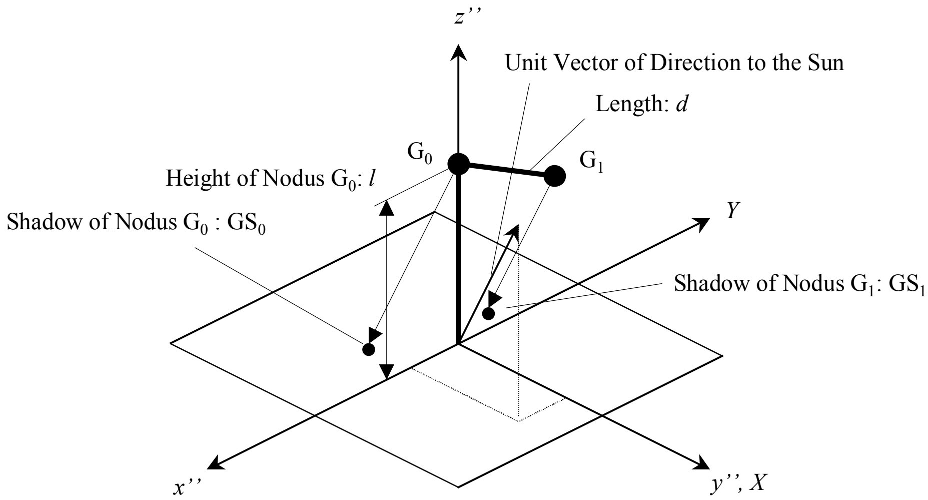

Figure 5. Shadows of Nodus

Finally, the sundial coordinate system is transformed to O-*X*-*Y* coordinate system in figure 5.

$$\begin{pmatrix} X_{GS0} \\ Y_{GS0} \\ Z_{GS0} \end{pmatrix} = \begin{pmatrix} y_{GS0}'' \\ -x_{GS0}'' \\ z_{GS0}'' \end{pmatrix}$$

$$\begin{pmatrix} X_{GS1} \\ Y_{GS1} \\ Z_{GS1} \end{pmatrix} = \begin{pmatrix} y_{GS1}'' \\ -x_{GS1}'' \\ z_{GS1}'' \end{pmatrix}$$

6. Sample Calculations

I made MS-Excel worksheets to calculate the lines of a sundial. You input data of your latitude and orientation of the dial face, then the worksheets calculate the positions of the shadows of the nodus on the dial face. Three examples are shown below.

6.1 Vertical Sundial to be used in Nagoya, Japan (Latitude 35°10' North)

A vertical sundial for Nagoya, Japan (35°10') which is facing south is calculated by the worksheet. The dial face of the vertical sundial is shown in figure 6.

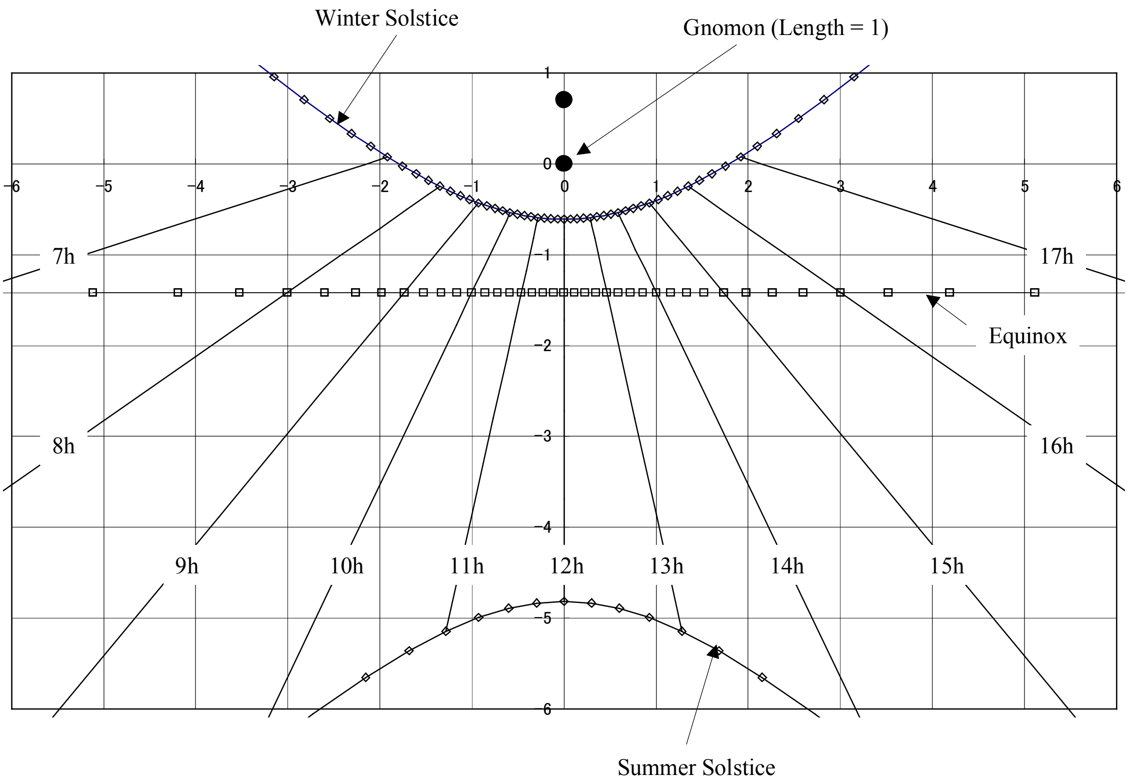

Figure 6. Vertical Sundial for Nagoya

6.2 Horizontal Sundial in Nagoya, Japan

Figure 7 shows a dial face of a horizontal sundial for Nagoya calculated by the worksheets.

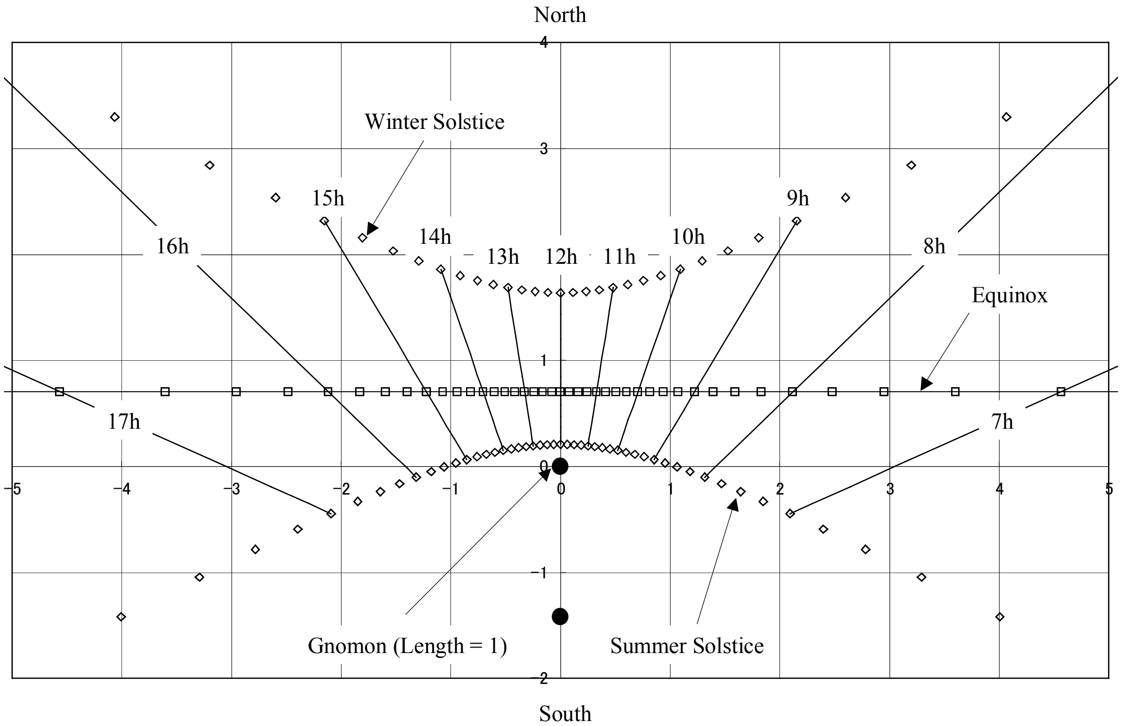

Figure 7. Horizontal Sundial for Nagoya

6.3 Sundial of Trinia dei Monti in Rome

Trinita dei Monti (figure 8), which is located above Spanish Steps and Piazza di Spagna in Rome, has a sundial on its façade. The latitude of Rome is 41°54'. The façade is facing 75° west from the south. I calculated the dial of the vertical sundial with these conditions using the worksheets. The result is shown in figure 9. The figure also shows the actual sundial for comparison.

Figure 8. Trinita dei Monti

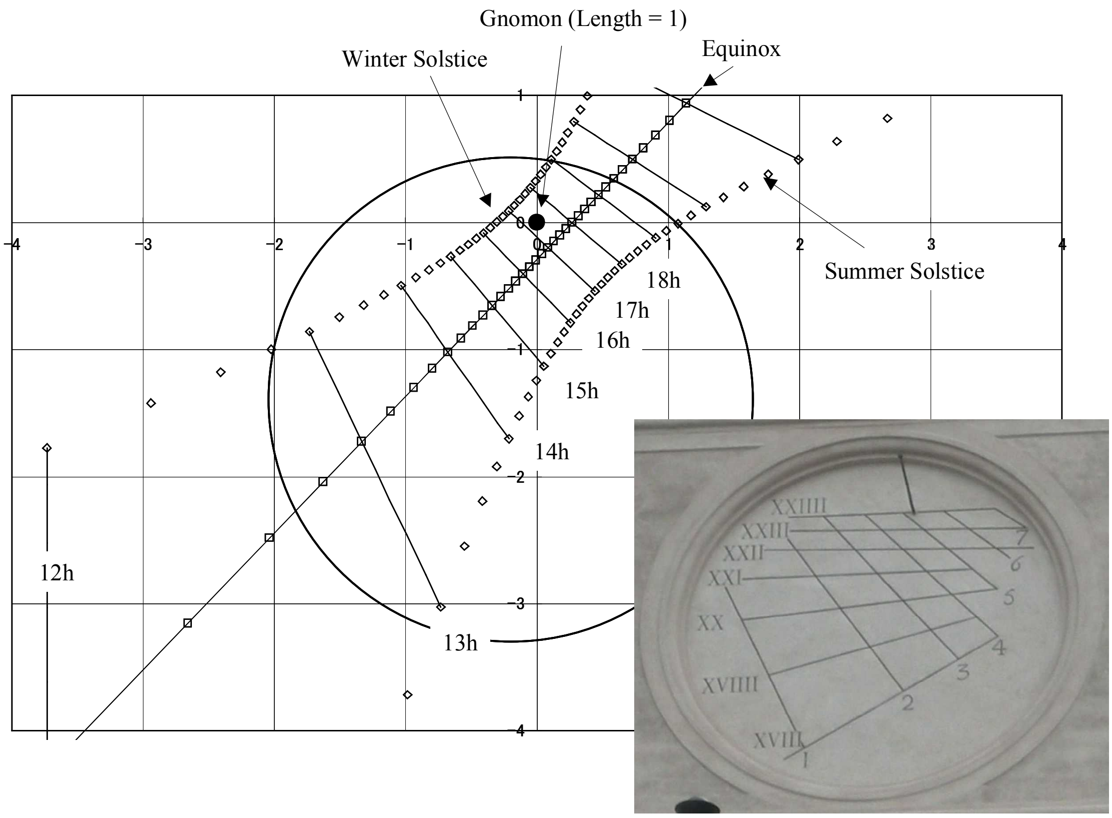

Figure 9. Vertical Sundial of Trinita dei Monti